Note on singular Clairaut-Liouville metrics

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Singular Clairaut-Liouville metrics. Let R be a meromorphic function such that (i) R and R' are positive on $[0,1[$, (ii) $X=1$ is a pole of finite order. Typically,

$$R = \sum_{n=0}^{p} \frac{a_n}{(1-X)^n}$$

with nonnegative a_n for $n = 0, p-1$, and positive a_p (p is the order). Let G denote the function $XR \circ \sin^2$, that is

$$G(\varphi) = R(\sin^2\varphi)\sin^2\varphi.$$

Clearly, G has the equatorial symmetry $G(\pi-\varphi) = G(\varphi)$.

Proposition 1. *The two vector fields*

$$F_1 = \frac{1}{\sqrt{G(\varphi)}}\frac{\partial}{\partial\theta} \quad \textit{and} \quad F_2 = \frac{\partial}{\partial\varphi}$$

define a complete sub-Riemannian[1] *metric on* $\mathbf{S}^2$.

Proof. Readily, $(\mathrm{ad}^k F_2)F_1 = (1/\sqrt{G})^{(k)}\partial/\partial\theta$. Now, around $\varphi = \pi/2$,

$$\frac{1}{\sqrt{G}} = O(\theta(X)),$$

with $\theta = (1-X)^{p/2} \circ \sin^2$, and

$$\theta^{(k)} = O((1-X)^{(p-k)/2} \circ \sin^2)$$

around $\pi/2$ again. So $\theta^{(p)}(\pi/2) \neq 0$ and the distribution generated by F_1 and F_2 verifies the Hörmander condition on the compact manifold $\mathbf{S}^2$. □

Remark 1. The finiteness assumption is crucial herebefore: for $R = \exp(1/(1-X)^2)$ which has an infinite order pole at $X=1$, one can check that $(1/\sqrt{G})^{(k)}$ is zero at $\varphi = \pi/2$ for any order of differentiation k, so that the Lie algebra generated by F_1 and F_2 is $\mathbf{R}\,\partial/\partial\varphi$ which has not full dimension.

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[1] Carnot-Carathéodory.

The associated metric on $\mathbf{S}^2$

$$g = G(\varphi)\mathrm{d}\theta^2 + \mathrm{d}\varphi^2$$

is called a *singular Clairaut-Liouville metric.*

Discrete symmetry group. We denote $\Gamma = 1/G$ the inverse of G, well defined then on $]0, \pi/2[$. We restrict to the level set $H = 1/2$ of the Hamiltonian associated to the metric,

$$H = \frac{1}{2}(\Gamma(\varphi)p_\theta^2 + p_\varphi^2),$$

and so parameterize extremals by arc length. Consider the extremal departing from $\varphi_0 \neq 0\,(\pi)$ (not a pole), θ_0 being normalized to 0 (cyclic variable) and defined by a positive p_θ (the degenerate case $p_\theta = 0$ corresponding to meridians—which are the only extremals passing through the poles) and non-negative $p_{\varphi_0} = \sqrt{1 - \Gamma(\varphi_0)p_\theta^2}$. Along the extremal, $\dot\varphi$ first vanishes when $\varphi = \pi - G^{-1}(p_\theta^2)$, the reciprocal function of G being well defined because of assumption (ii) and since

$$G' = (R + XR') \circ \sin^2 \cdot 2 \sin\cos.$$

As neither G' nor Γ' vanish at $G^{-1}(p_\theta^2) \in]0, \pi/2[$,

$$1 - \Gamma(\varphi)p_\theta^2 = O((\pi - G^{-1}(p_\theta^2)) - \varphi)$$

in the neighbourhood of $\pi - G^{-1}(p_\theta^2)$, and the following integral is well-defined:

$$t_1(p_\theta, \varphi_0) = \int_{\varphi_0}^{\pi - G^{-1}(p_\theta^2)} \frac{\mathrm{d}\varphi}{\sqrt{1 - \Gamma(\varphi)p_\theta^2}}.$$

Lemma 1. *The axial symmetry with respect to $\theta_1 = \theta(t_1)$ leaves the extremal invariant.*

Proof. Set

$$\begin{array}{rclcrcl} \hat\theta &=& 2\theta_1 - \theta(2t_1 - t), & \qquad & \hat p_\theta &=& p_\theta, \\ \hat\varphi &=& \varphi(2t_1 - t), & & \hat p_\varphi &=& -p_\varphi(2t_1 - t), \end{array}$$

and check that new curve is still an extremal, passing through the same point of the cotangent bundle at t_1 since $p_\varphi(t_1) = 0$. $\square$

Necessarily, $\pi - G^{-1}(p_\theta^2) \geq \pi - \varphi_0$, so there also exists $t_2 \leq t_1$ such that $\varphi(t_2) = \pi - \varphi_0$. Using the previous axial symmetry, we deduce the existence of $t_3 = 2t_1 - t_2 \geq t_2$ such that, again, $\varphi(t_3) = \pi - \varphi_0$. Using now the equatorial symmetry of G (and $\Gamma = 1/G$), the following is clear.

Lemma 2. *The central symmetry with respect to $(\theta(t_3/2), \varphi(t_3/2))$ defines another extremal with same initial condition.*

Finally denote t_4 the point such that $\varphi(t_4) = \pi/2 \leq \pi - \varphi_0$, and remark that the central symmetry with respect to $(\theta(t_4), \varphi(t_4))$ leaves the extremal invariant. Since the axial symmetry with respect to $\theta = 0$ obviously defines another extremal with same initial condition, we conclude that the four-order group generated by one axial symmetry with respect to θ (denoted s_1) and one central symmetry (denoted s_2) acts on extremals with same initial condition, and also on extremals themselves (thus defining innner symmetries).

Proposition 2. *The Klein group acts on the set of extremals issuing from the same point, as well as on every extremal.*

An extremal is said to be a *pseudo-equator* whenever $\dot{\varphi}(0) = p_\varphi(0)$ is equal to zero.

Proposition 3. *Every extremal which is not a meridian is a pseudo-equator.*

Proof. For p_θ positive and $p_{\varphi 0}$ nonnegative (the other cases are deduced by symmetry), there exists $\tilde{\varphi}_0 = G^{-1}(p_\theta^2)$ such that, up to a time shift, the extremal is the pseudo-equator with initial condition $\tilde{\varphi}_0$. □

Corollary 1. *On every extremal, the φ coordinate is periodic with period*

$$T(p_\theta) = 4\int_{G^{-1}(p_\theta^2)}^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-\Gamma(\varphi)p_\theta^2}}\,,$$

and $\theta(t+T) = \theta(t) \pm \Delta\theta$ (the sign depending on the sign of p_θ) with

$$\Delta\theta(p_\theta) = 4\int_{G^{-1}(p_\theta^2)}^{\pi/2} \frac{\Gamma(\varphi)p_\theta \mathrm{d}\varphi}{\sqrt{1-\Gamma(\varphi)p_\theta^2}}\cdot$$

Proof. According to the previous analysis, it is enough to check the result on pseudo-equators. But then, $t_1 = t_2 = t_3 = 2t_4$, so setting $T = 2t_1$ and using the axial symmetry with respect to θ_1 gives the result since $\varphi(T) = \varphi(0)$, $p_\varphi(T) = -p_\varphi(0) = 0 = p_\varphi(0)$. Hence $\dot{\theta} = \Gamma(\varphi)p_\theta$ is also periodic which concludes the proof. □

Remark 2. Since G defines a one-to-one (strictly increasing) mapping between $]0, \pi/2[$ and $\mathbf{R}_+^*$, we can as before consider that extremals are not parameterized by their Clairaut constant, p_θ, but rather by their initial condition $\varphi_0 = G^{-1}(p_\theta^2)$ as pseudo-equators. Then,

$$T(\varphi_0) = 4\int_{\varphi_0}^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-\Gamma(\varphi)/\Gamma(\varphi_0)}}\,,$$

and

$$\Delta\theta(\varphi_0) = 4\int_{\varphi_0}^{\pi/2} \frac{\Gamma(\varphi)\mathrm{d}\varphi}{\sqrt{\Gamma(\varphi_0)-\Gamma(\varphi)}}\cdot$$

These relations actually cover the case of meridians $\varphi_0 = 0$ (*i.e.* $p_\theta = 0$) for which $T = 2\pi$ and $\Delta\theta = 2\pi$ (two instantaneous rotations of angle π when crossing poles at $t = \pi$ and $t = 2\pi$). Both T and $\Delta\theta$ vanish when $\varphi_0 = \pi/2$ since extremals accumulate at the initial point as will be clear using the following local model.

Quasi-homogeneous local model. Setting $x = \pi/2 - \varphi$ and $y = \theta$, a local model at $\varphi = \pi/2$ is

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \frac{\mathrm{d}y^2}{x^{2p}}$$

where p is the order of the pole. The equatorial symmetry of G is approximated by $1/(-x)^{2p} = 1/x^{2p}$, so the discrete symmetry group is preserved. One gets [3]

$$x = \frac{1}{\sqrt[p]{\lambda}} q(t\sqrt[p]{\lambda}), \quad y = \frac{1}{(\sqrt[p]{\lambda})^{p+1}} r(t\sqrt[p]{\lambda}),$$

for extremals departing from the origin, where q and r are hypergeometric functions depending on p, and denoting $\lambda = p_y > 0$. The conjugate locus is the set of first critical values of the exponential mapping on $H = 1/2$,

$$\exp_t(\lambda) = \exp_{(0,0),t}(\lambda) = (x(t,\lambda), y(t,\lambda)),$$

so that, because of quasi-homogeneity, conjugate times are $t_{1c}(\lambda) = s_p/\sqrt[p]{\lambda}$ where s_p is a root of

$$qr' - (p+1)q'r = 0.$$

Accordingly, the conjugate locus is described as follows [3].

Lemma 3. *The conjugate locus at the origin of* $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2/x^{2p}$ *is the set* $y = \pm C_p x^{p+1}$ *minus the origin where*

$$C_p = \frac{1}{p+1}\sqrt{\frac{q^{2p}(s_p)}{1 - q^{2p}(s_p)}}.$$

Cut and conjugate loci.

Proposition 4. *A cut point of the singular Clairaut-Liouville metric is either a conjugate point, or a point in the separating line.*

Proof. First assume $\varphi_0 \neq \pi/2$. If the cut point is not a conjugate point, the exponential mapping is a diffeomorphism in the neighbourhood of the time, t_l, and adjoint vector that generate the cut point. Since the metric is complete by Proposition 1, there exist minimizing extremals γ_n joining the initial point to $\gamma(t_l + 1/n)$, $n \geq 1$, where γ is the curve in the state space defining the cut point, $\gamma(t_l)$. As $\varphi_0 \neq \pi/2$, the set $\{p = (p_\theta, p_\varphi) \,|\, H(0, \varphi_0, p) = 1/2\}$ is compact, and one can extract a converging subsequence of the $(p_n)_n$ generating the extremals γ_n, and thus get the standard contradiction [4].

When $\varphi_0 = \pi/2$, though $\{p \,|\, H = 1/2\} = \mathbf{R} \times \{p_\varphi = \pm 1\}$ is not bounded anymore, $(p_{\theta n})_n$ still has to be bounded otherwise there would exist a subsequence such that $|p_{\theta n}| \to \infty$, and $\gamma_n(t_l + 1/n)$ would tend to $(0, \pi/2)$ (as is clear from an estimation of T and $\Delta\theta$ when $p_\theta \to \infty$), the initial point, not to the cut point $\gamma(t_l)$. The sequence being thus bounded, we can conclude as before. □

We introduce the new assumption on the metric that (iii) $\Delta\theta$ is strictly decreasing.

Theorem 1. *The cut locus of any point is simple and antipodal. More precisely, the cut locus of a pole is reduced to the opposite pole, is equal to the equator minus the point itself when* $\varphi_0 = \pi/2$*, to a proper closed subarc of the antipodal parallel otherwise.*

Proof. The case of poles is obvious since the only extremals through them are meridians.

Consider now the situation $\varphi_0 = \pi/2$, and show that the exponential mapping is injective on the quadrant

$$D = \cup_{p_\theta>0}[0, T(p_\theta/2)] \times \{p_\theta, 1\},$$

that is show that subarcs of extremals defined by $t \in [0, T(p_\theta)/2]$, positive p_θ and $p_\varphi = +1$ do not intersect. If $p'_\theta > p_\theta$, the arc associated with p'_θ is strictly below the one associated with p_θ. Indeed, note that on the first half of such an arc ($t \in [0, T/4[$), $\dot\varphi$ does not vanish so that the curve can be parameterized by φ. There,

$$f(\varphi, p_\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}\varphi} = \frac{\Gamma(\varphi)p_\theta}{\sqrt{1-\Gamma(\varphi)p_\theta^2}}\,,$$

is an increasing function of p_θ since

$$\frac{\partial f}{\partial p_\theta} = \frac{\Gamma(\varphi)}{(1-\Gamma(\varphi)p_\theta^2)^{3/2}} > 0.$$

As geodesics starting from $\varphi_0 = \pi/2$ cross again the equator at $\Delta\theta/2$, assumption (iii) ensures that the aforementioned subarcs do not intersect. We conclude by remarking that the full set of extremals is obtained by considering the action of the Klein group: first, the central symmetry s_2 which generates intersections at $t = T/2$, then the axial symmetry s_1 with respect to $\theta = 0$ which generates intersections at $\theta = \pi$, thus not prior to the previous ones since $\theta(T/2) = \Delta\theta/2$, and since $\Delta\theta < 2\pi$ for $p_\theta > 0$ (by assumption, $\Delta\theta$ is decreasing, and equal to 2π on meridians, *i.e.* when $p_\theta = 0$). So extremals are optimal up to $t = T/2$, and the corresponding point belongs to the separating line. Since the metric is complete, each point of the equator is reached by such an extremal and the separating line, hence the cut locus, is the equator minus the initial point itself.

Consider finally the case when the initial point is neither a pole nor on the equator. Then, p_θ^2 belongs to $]0, G(\varphi_0)[$, and extremals are again optimal up to $t = T/2$. Indeed, there would otherwise exist shorter extremals which would lead to the existence of shorter extremals for the initial condition $\varphi_0 = \pi/2$, too, contradicting the previous fact. The central symmetry s_2 still generates an intersection at $t = T/2$, and $\varphi(T/2) = \pi - \varphi_0$ so the corresponding point in the separating line belongs to the antipodal parallel of the starting point. Since $\Delta\theta$ is decreasing, the extremities of the cut are obtained letting p_θ tend to $\pm\sqrt{G(\varphi_0)}$ (now finite, since $\varphi_0 \neq \pi/2$), and the subarc is closed. □

To get the result on the conjugate locus, we finally assume that (iv) $\Delta\theta$ is convex.

Theorem 2. *The conjugate locus of a point on the equator is double-hearted (four meridional cusps), astroidal otherwise (two meridional and two equatorial cusps).*

Proof. The analysis outside the equator being a direct extension of [2] result, we focus on the proof for $\varphi_0 = \pi/2$. Consider an extremal defined by a positive p_θ and $p_{\varphi_0} = +1$. For t in $]T/4, 3T/4[$, $\dot\varphi \neq 0$ and the extremal can be parameterized by φ according to

$$\theta(\varphi, p_\theta) = \frac{\Delta\theta(p_\theta)}{2} + \int_\varphi^{\pi/2} f(\varphi', p_\theta)\mathrm{d}\varphi',$$

where, as before,

$$f(\varphi, p_\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}\varphi} = \frac{\Gamma(\varphi)p_\theta}{\sqrt{1-\Gamma(\varphi)p_\theta^2}}.$$

The conjugacy condition writes $\partial\theta/\partial p_\theta = 0$, so $\varphi_{1c}(p_\theta)$ is solution of

$$\int_{\varphi_{1c}(p_\theta)}^{\pi/2} \frac{\partial f}{\partial p_\theta}(\varphi', p_\theta)\mathrm{d}\varphi' = -\frac{\Delta\theta'(p_\theta)}{2} > 0,$$

in order that $\varphi_{1c}(p_\theta) < \pi/2$ (for $\partial f/\partial p_\theta > 0$). By differentiating the previous equality, one gets

$$\varphi_{1c}(p_\theta) = \left[\frac{\partial f}{\partial p_\theta}(\varphi_{1c}(p_\theta), p_\theta)\right]^{-1} \left[\frac{\Delta\theta''}{2}(p_\theta) + \int_{\varphi_{1c}(p_\theta)}^{\pi/2} \frac{\partial^2 f}{\partial p_\theta^2}(\varphi', p_\theta)\mathrm{d}\varphi'\right],$$

which is positive because $\varphi_{1c}(p_\theta) < \pi/2$, because

$$\frac{\partial^2 f}{\partial p_\theta^2} = \frac{3\Gamma^2(\varphi)p_\theta}{(1-\Gamma(\varphi)p_\theta^2)^{5/2}} > 0,$$

and by virtue of (iv) (positiveness of $\Delta\theta''$). In particular, the parameterization $p_\theta \mapsto (\varphi_{1c}(p_\theta), \theta(\varphi_{1c}(p_\theta), p_\theta))$ of the conjugate locus is regular, and the locus has no cusp for positive p_θ.

The tangent vector along the conjugate locus is

$$\left(\varphi_{1c}' \cdot \frac{\partial\theta}{\partial\varphi} + \underbrace{\frac{\partial\theta}{\partial p_\theta}}_{0}\right)\frac{\partial}{\partial\theta} + \varphi_{1c}' \cdot \frac{\partial}{\partial\varphi},$$

proportional to $f(\varphi_{1c}(p_\theta), p_\theta)\partial/\partial\theta + \partial/\partial\varphi$. On the one hand, as in the regular case discussed in [2], $f(\varphi_{1c}(p_\theta), p_\theta)$ tends to 0 when $p_\theta \to 0+$, and the locus has a first meridional cusp because of the axial symmetry s_1. On the other hand, when p_θ goes to $+\infty$, the analysis on the local model shows that, contrary to the regular case, there is a second meridional cusp at the initial point where conjugate points accumulate, and we have a heart-shaped conjugate locus. The central symmetry s_2 gives the symmetric part, hence the result. □

The results directly apply to the simplest case of order one, $R = 1/(1-X)$, associated with the bi-entry orbit transfer. Indeed, one has $\Delta\theta(\varphi_0) = 2\pi(1 - \sin\varphi_0)$ (see [1], and compare with the—doubled—*half period function* when λ goes to ∞ in [2]), that is $\Delta\theta(p_\theta) = 2\pi(1 - p_\theta/\sqrt{1+p_\theta^2})$, so that

$$\Delta\theta' = -\frac{1}{(1+p_\theta^2)^{3/2}} < 0, \quad \Delta\theta'' = \frac{3p_\theta}{(1+p_\theta^2)^{5/2}} > 0.$$

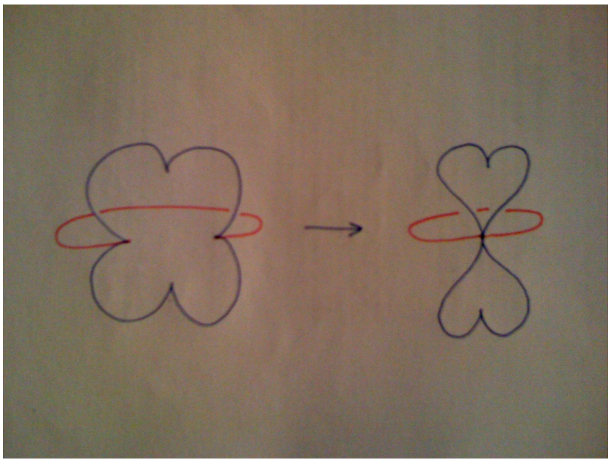

Figure 1: Bifurcation of the conjugate locus (in blue) from an astroid—that is a quatrefoil, as seen from behind—to a double-heart when the initial condition goes to the equator. The same bifurcation occurs on the homotopy regularizing the metric described in [3] when λ goes to one. The cut locus (in red) is an open arc for a point on the equator.

References

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